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# Nonassociative Geometry and Discrete Space-Time

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A new mathematical theory, *nonassociative geometry*, is surveyed and a research program for studying the discrete structure of space-time is proposed. Nonassociative geometry gives a unified algebraic description of smooth, continuous, and discrete space-time.

## 1. WHAT IS NONASSOCIATIVE GEOMETRY?

The recent development of geometry has shown the importance of nonassociative algebraic structures such as quasigroups, loops, and odules. The nonassociativity is the algebraic equivalent of the curvature. In a neighborhood of an arbitrary point on a manifold with an affine connection, one can introduce the geodesic local loop, which is uniquely defined by means of the parallel translation of geodesics along geodesics (Kikkawa, 1964; Sabinin, 1977). The family of local loops constructed in this way uniquely defines the space with affine connection, but not every family of geodesic loops on a manifold defines an affine connection. It is necessary to add algebraic identities connecting loops in different points. Sabinin (1977, 1981) introduced *geoodular structures* and showed the equivalence of the categories of geoodular structures and affine connections. The main algebraic structures arising are related to nonassociative algebra and the theory of quasigroups and loops. Here we survey the algebraic foundations of *nonassociative geometry* due to Sabinin (1972a, b, 1977, 1981, 1988, 1989, 1994, 1999; Sabinin and Nesterov, 1997).

*1.1. Definition (Quasigroup and loop; Richard H. Bruck, 1971, Valentin, D. Belousov, 1967).* Let $Q$ be a set with a binary operation $(a, b) \mapsto a \cdot b$,

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$\forall a, b \in Q$. Then $\langle Q, \cdot \rangle$ is said to be a magma (groupoid). A groupoid $\langle Q, \cdot \rangle$ is called a *quasigroup* if the equations $a \cdot x = b$, $y \cdot a = b$ have unique solutions $x = a \backslash b$, $y = b/a$. A *loop* is a quasigroup with a two-sided identity $\varepsilon \in Q$, $a \cdot \varepsilon = \varepsilon \cdot a = a$. See also Sabinin (1999).

If $Q$ is a smooth manifold, then a loop $\langle Q, \cdot, \varepsilon \rangle$ with a smooth function $\varphi(a, b) = a \cdot b$ is called a *smooth loop*.

*1.2. Definition* (Local loop; Kikkawa, 1964). Let $\langle M, \cdot, \varepsilon \rangle$ be a partial magma with a binary operation $(x, y) \mapsto x \cdot y$ and the neutral element $\varepsilon$, $x \cdot \varepsilon = \varepsilon \cdot x = x$. Let $M$ be $C^1$-smooth manifold and the $C^1$-smooth operation of multiplication be defined in some neighborhood $U_g$ of $\varepsilon$. Then $\langle M, \cdot, \varepsilon \rangle$ is called a *local loop* on $M$.

Let $\langle Q, \cdot, \varepsilon \rangle$ be a smooth local loop with a neutral element $\varepsilon$. We define a left translation $L_a$, a right translation $R_b$, and an associator $l_{(a,b)}$,

$$L_a b = R_b a = a \cdot b, \qquad l_{(a,b)} = (L_{a \cdot b})^{-1} \circ L_a \circ L_b$$

The operation of multiplication is locally left and right invertible; if $x \cdot y = L_x y = R_y x$, then there exist $L_x^{-1}$ and $R_x^{-1}$ in some neighbourhood of the neutral element $\varepsilon$, $L_a(L_a^{-1} x) = x$, $R_a(R_a^{-1}\, x) = x$.

The left basic fundamental vector fields $A_j$ and the right basic fundamental vector fields $a_j$ are defined on $U_g$ by

$$A_j(x) = ((L_x)_{*,\varepsilon})^i_j \frac{\partial}{\partial x^i} = A^i_j(x) \frac{\partial}{\partial x^i}$$

$$a_j(x) = ((R_x)_{*,\varepsilon})^i_j \frac{\partial}{\partial x^i} = a^i_j(x) \frac{\partial}{\partial x^i}$$

The equation $df^i(t)/dt = A^i_j(f((t))X^j$, $f(0) = \varepsilon$, has the solution $f(t) = \operatorname{Exp} tX$ and defines the left exponential map

$$\operatorname{Exp}: \quad T_\varepsilon(M) \ni X \mapsto \operatorname{Exp} X \in M$$

For $t \in \mathbb{R}$, the operation $M \ni x \mapsto tx = \operatorname{Exp}(t \operatorname{Exp}^{-1} x) \in M$ is called the *left canonical unary operation* for $\langle M, \cdot, \varepsilon \rangle$. A smooth loop $\langle M, \cdot, \varepsilon \rangle$ with its canonical left unary operations is called the *canonical left preodule* $\langle M, \cdot, (t)_{t \in \mathbb{R}}, \varepsilon \rangle$. If

$$x + y = \operatorname{Exp}(\operatorname{Exp}^{-1} x + \operatorname{Exp}^{-1} y)$$

then we obtain the *canonical left prediodule* of a loop, $\langle M, \cdot, +, (t)_{t \in \mathbb{R}}, \varepsilon \rangle$. A canonical left preodule (prediodule) is called the *left odule* (*diodule*) if the *monoassociativity* and **quasiassociativity** properties are satisfied,

$$t, u \in \mathbb{R}: \qquad tx \cdot ux = (t + u)x, \quad t(ux) = (tu)x$$

In the smooth case, for an odule, the left and the right canonical operations as well as the exponential maps coincide.

*1.3. Definition* (Loopuscular structure). Let $M$ be a smooth manifold and $L$: $M \ni (x, y, z) \mapsto L(x, y, z) \in M$ be a smooth partial ternary operation, such that $x \underset{a}{\cdot} y = L(x, a, z)$ defines in some neighborhood of the point $a$ the loop with the neutral $a$. Then the pair $\langle M, L\rangle$ is called a *loopuscular structure* (*manifold*).

A smooth manifold $M$ with a smooth partial ternary operation $L$ and smooth binary operations $\omega_t$: $(a, b) \in M \times M \mapsto \omega_i(a, b) = t_a b \in M$ $(t \in \mathbb{R})$ such that $x \underset{a}{\cdot} y = L(x, a, y)$ and $t_a z = \omega_t(a, z)$ determine in some neighborhood of an arbitrary point $a$ the odule with the neutral element $a$ is called a *left odular structure* (*manifold*) $\langle M, L, (\omega_t)_{t\in\mathbb{R}}\rangle$. Let $\langle M, L, (\omega_t)_{t\in\mathbb{R}}\rangle$ and $\langle M, N, (\omega_t)_{t\in\mathbb{R}}\rangle$ be odular structures; then $\langle M, L, N, (\omega_t)_{t\in\mathbb{R}}\rangle$ is called a *diodular* structure (manifold). If $x \underset{a}{+} y = N(x, a, y)$ and $t_a x = \omega_t(a, x)$ define a vector space, then such a diodular structure is called a *linear diodular structure*. A diodular structure is said to be *geodiodular* if the following hold:

$$L^{t_a x}_{u_a x} \circ L^a_{t_a x} = L^a_{u_a x} \quad (L^a_x y = L(x, a, y)) \qquad \text{first geoodular identity}$$

$$L^a_x \circ t_a = t_x \circ L^a_x \qquad \text{second geoodular identity}$$

$$L^a_x N(y, a, z) = N(L^a_x y, x, L^a_x z) \qquad \text{third geoodular identity}$$

*1.4. Definition.* Let $M$ be a $C^k$-smooth $(k \geq 3)$ affinely connected, manifold, $\mathrm{Exp}_x$ be the exponential map at the point $x$, and $\tau^a_x$ be the parallel translation along the geodesic going from $a$ to $x$. Let the following operations be given on $M$:

$$L^a_x y = x \underset{a}{\cdot} y = \mathrm{Exp}_x\, \tau^a_x\, \mathrm{Exp}^{-1}_a y$$

$$\omega_t(a, z) = t_a z = \mathrm{Exp}_a\, t\, \mathrm{Exp}^{-1}_a z$$

$$N(x, a, y) = x \underset{a}{+} y = \mathrm{Exp}_a(\mathrm{Exp}^{-1}_a x + \mathrm{Exp}^{-1}_a y)$$

The construction above equips $M$ with the linear geodiodular structure which is called a *natural linear geodiodular structure of an affinely connected manifold* $(M, \nabla)$.

Any $C^k$-smooth $(k \geq 3)$ affinely connected manifold can be considered as a geodiodular structure.

*1.5. Definition.* Let $\langle M, L\rangle$ be a loopuscular structure of a smooth manifold $M$ and $Y$ be a vector field in the neighborhood of a point $a$. Then the following formulas define the *tangent affine connection* and its components:

$$\nabla_{X_a} Y = \left\{ \frac{d}{dt} ([(L^a_{g(t)})^*, a]^{-1} Y_{g(t)}) \right\}_{t=0}, \qquad g(0) = a, \quad \dot{g}(0) = X_a$$

$$\Gamma^i_{jk}(a) = -\left[ \frac{\partial^2 (x \underset{a}{\cdot} y)^i}{\partial x^j \partial y^k} \right]_{x=y=a}$$

Sabinin (1977, 1981) showed the equivalence of the categories of geoodular (geodiodular) structures and of affine connections.

*1.6. Definition* (Elementary holonomy). Let $\langle M, L \rangle$ be a loopuscular structure. Then

$$h^a_{(b,c)} = (L^a_c)^{-1} \circ L^b_c \circ L^a_b$$

is called an *elementary holonomy.*

*1.7. Comment.* An elementary holonomy is the parallel translation along a geodesic triangle path; it is some integral curvature. In the smooth case, differentiating $(h^a(x, y))^i_p$ by $x^j$, $y^k$ at $a \in M$, we get the curvature tensor at $a \in M$ up to a numerical factor.

*1.8. Comment.* For a diodular structure, one can consider an elementary holonomy $h_{(a,b)} = h^{\varepsilon}_{(a,b)}$ together with the diodule $\langle M, \underset{\varepsilon}{\cdot}, \underset{\varepsilon}{+}, (t_\varepsilon)_{t \in \mathbb{R}}, \varepsilon \rangle$, a so-called *holonomial diodule*, and restore this diodular structure in a unique way:

$$L(x, a, y) = L^a_x h_{(a,x)} (L^{\varepsilon}_a)^{-1} y$$

$$N(x, a, y) = L^{\varepsilon}_a ((L^{\varepsilon}_a)^{-1} x \underset{\varepsilon}{+} (L^{\varepsilon}_a)^{-1} y)$$

$$\omega_t(a, y) = t_a y = L^{\varepsilon}_a t_\varepsilon (L^{\varepsilon}_a)^{-1} y$$

In this case the holonomial identities hold,

linearity: $h_{(a,b)} t_\varepsilon x = t_\varepsilon h_{(a,b)} x, \qquad h_{(a,b)}(x \underset{\varepsilon}{+} y) = h_{(a,b)} x \underset{\varepsilon}{+} h_{(a,b)} y$

joint identity: $h_{(a, a \cdot u_\varepsilon b)} tb = l_{(a, u_\varepsilon b)} tb$

*h*-identity: $h_{(c \cdot t_\varepsilon a, c \cdot u_\varepsilon a)} h_{(c, c \cdot t_\varepsilon a)} x = h_{(c \cdot u_\varepsilon a)} x$

ε-identity: $h_{(\varepsilon, q)} x = x$

*1.9. Comment.* The generalized Bianchi identities hold:

$$h^a_{(z,x)} \circ h^a_{(y,z)} \circ h^a_{(x,y)} = (L^a_x)^{-1} \circ h^x_{(y,z)} \circ L^a_x \tag{1}$$

In the linear approximation, (1) reestablishes the usual Bianchi identities.

Nonassociative geometry is based on the constructions described above and may be applied to nonsmooth, discrete, or even finite geometry.

## 2. NONASSOCIATIVE DISCRETE GEOMETRY

In this section, we present a research program in a new direction rather than a ready-made physical theory. There are no physical results yet. We are convinced that discrete geometry useful in physics has to have the main features of the established smooth geometry. We propose to use geoodular (geodiodular) spaces as algebraic models for affinely connected spaces in the nonsmooth case. We consider finite affinely connected spaces. This dictates use of a finite field $\mathcal{F} = \langle F, +, -(\ ), \cdot, 0, 1\rangle$ instead of reals $\mathbb{R}$. There are some variants where $\mathcal{F}$ is a ring or double loop, etc. In this paper, for simplicity, we consider the case where $\mathcal{F}$ is a finite field. The finite fields are Galois fields $\mathcal{GF}(p^m)$, $p$ being a prime number, $m = 1, 2, \ldots$; in this case, $p^m$ is the number of elements of the field and $x^{p^m} = x$.

*2.1.* Consider a geodiodular finite space $\mathcal{M} = \langle M, L, N, (\omega_t)_{t\in F}\rangle$ with ternary operations $L(x, y, z) = x \underset{y}{\dot{+}} z$, $N(x, y, z) = \dfrac{x + z}{y}$ and a system of binary operations $\omega_t(x, y) = t_x y$ $(t \in F)$:

1. $\langle M, \underset{a}{\dot{+}}, (t_a)_{t\in F}\rangle$ is an odule with a neutral $a \in M$ for any $a \in M$.
2. $\langle M, \underset{a}{+}, (t_a)_{t\in F}\rangle$ is an $n$-dimensional vector space (with zero $a \in M$).
3. The following geoodular identities hold:

$$\text{first:} \qquad L^{t_a x}_{u_a x} \circ L^{a}_{t_a x} = L^{a}_{u_a x} \qquad (L^a_x\, y = L(x, a, y))$$

$$\text{second:} \qquad L^a_x \circ t_a = t_x \circ L^a_x$$

$$\text{third:} \qquad L^a_x(y \underset{a}{+} z) = L^a_x y \underset{x}{+} L^a_x z$$

If needed, we may consider the operations above as partially defined.

*2.2.* In our approach, $\mathcal{M}^+_a = \langle M, \underset{a}{+}, a, (t_a)_{t\in F}\rangle$ plays the role of tangent space at $a \in M$ because in the classical smooth case, the tangent space $T_a M$ to $a \in M$ may be identified, at least locally, with $M$ by means of the exponential map. For $\forall a \in M$, any $x \in M$ may be regarded as a vector in $\mathcal{M}^+_a$.

Any line $(t_a b)_{t\in F}$ may be regarded as a geodesic through $a, b \in M$. As to parallelism, $y \mapsto b \underset{a}{\dot{+}} y$ means that the vector $b \underset{a}{\dot{+}} y$ (of $\mathcal{M}^+_b$) is parallel to a vector $y$ (of $\mathcal{M}^+_a$) along the geodesic $(t_a b)_{t\in F}$.

The presence of curvature results in nontrivial (in general) elementary holonomy,

$$L^y_a \circ L^x_y \circ L^a_x = h^a(x, y) \neq \mathrm{id}$$

Thus $h^a(x, y) = \mathrm{id}$ for any $a, x, y, \in M$ means absolute parallelism of our space and if, additionally, $x \underset{a}{+} y = x \underset{a}{\dot{+}} y$ (or $L = N$, which is the same), it means the absence of torsion, that is, we come to a classical affine space (Sabinin, 1999, Ch. 3).

*2.3. Remark.* We do not know, in general, how the absence of torsion can be expressed in purely algebraic terms. There are some sufficient conditions (Sabinin, 1999).

*2.4.* One can introduce the notion of a curve in discrete space as an ordered set of points (perhaps not all different) $\gamma = (a_1, a_2, \ldots, a_s)$ and, further, define the vector $y$ (of $\mathcal{M}^+_{a_s}$) parallel to a vector $x$ (of $\mathcal{M}^+_{a_1}$) along this curve as

$$y = L^{a_{s-1}}_{a_s} \cdots L^{a_2}_{a_3} L^{a_1}_{a_2}\, x \qquad (L^a_b x \overset{\text{def}}{=} b \dot{\underset{a}{}} x)$$

*2.5. Remark.* One can introduce the holonomy group $\mathcal{H}_a\mathcal{M}$ at a point $a \in M$ considering the parallel translations along all curves with coinciding initial and final point. Any element of $\mathcal{H}_a\mathcal{M}$ is a composition of elementary holonomies at $a \in M$.

*2.6. Metric* (Sabinin, 1999). Having given a finite diodular space (finite affinely connected space, in other words), we define a *metric diodular structure*, i.e., a nondegenerate, symmetric, twice covariant tensor $g_a(x, y)$ for any $\mathcal{M}^+_a$. The condition of covariant permanence is

$$g_b(L^a_b x, L^a_b y) = g_a(x, y)$$

*2.7. Remark.* Any finite field consists of $p^m$ elements, $p$ is prime, $m \in \mathbb{N}$. In order to apply the above construction to gravity, we need to classify spaces with small $p = 2, 3, 5, 7, 11$ and $m = 1, 2, 3, 4$ such that dim $\mathcal{M}^+_a \leq 4$. Such a classification is significant for discrete physical models of space-time.

*2.8.* For the description of the evolution of space-time, the hypothesis of increasing number of points is quite natural. For example, one may consider space-time as 'glued' from three-dimensional spaces with different $p$, $m$. This means that we should regard $p$ and $m$ as growing in time. Probably we need some additional physical reasons to assume the 'true' law of dependence of $p$ and $m$ on the time.

We should invent a limit procedure in order to obtain the usual smooth schemes of space-time approaching time to infinity.

## 3. THE CALCULUS IN DISCRETE SPACES

Attempts to use finite differences for the discrete case are not very effective. Instead, we may use the observation that any function given on a direct finite product of finite fields

$$f\colon \underbrace{F \times \cdots \times F}_{l \text{ times}} \to F, \qquad \mathscr{F} = \langle F, +, -(\,), 0, \cdot, 1 \rangle$$

is a polynomial.

We should use the minimal degree polynomial representation [for example, for $\mathscr{GF}(p^m)$, $x^{p^m} = x$, that is, the minimal representation for $x^{p^m}$ is $x$]. Any polynomial $\mathrm{P}(x) = a_0 + a_1 x + \cdots + a_n x^n$ $(a_n \neq 0)$ is algebraically differentiable, $\mathrm{P}'(x) = a_1 + \cdots + (a_n\, n)x^{n-1}$, and similarly for polynomials with several unknowns.

One should be careful with the properties of such a calculus:

1. For $\mathscr{GF}(p^m)$, $(x^p)' = p(x^{p-1}) = 0$, $p$ being the characteristic.
2. The Leibniz formula $(fg)' = fg' + f'g$ is not valid in general.
3. There is no existence and uniqueness theorem for a solution of differential equation with the usual initial conditions.

In our case, we cannot work with infinitesimal objects only—the algebraic structure of a diodular space should be always taken into account.

The tangent space and vector fields can be introduced. Let $Y$ be a vector field and [ , ] be the Lie bracket. The affine connection $\nabla$ and the tensors of torsion and curvature are introduced as follows:

$$\nabla_{X_a} Y = \left\{ \frac{d}{dt} [(L^a_{\gamma(t)})^{-1}_{*,a} Y_{\gamma(t)}] \right\}_{t=0}, \qquad \gamma(0) = a, \quad \dot{\gamma}(0) = X_a$$

$$T(X, Y) = \nabla_X Y - \nabla_X Y - [X, Y]$$

$$R(X, Y) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

A space of zero torsion may be introduced.

One can write the Einstein equation including (or not) the torsion. Due to the absence of an existence and uniqueness theorem for differential equations in the usual sense, the solution (if any) has some properties which are not valid in the classical smooth case.

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